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Inertial convection at low Prandtl number

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The problem of Rayleigh–Bénard convection at low Prandtl number σ is investigated in a circular geometry. Jones, Moore & Weiss (1976) have formulated, but not solved analytically, an asymptotic nonlinear problem in the limit $\sigma \to 0$ at small velocities. It is shown that the problem they posed can be solved exactly in this geometry. The solutions are extended by means of expansions in the amplitude ϵ and the reciprocal of the Reynolds number $\sigma\epsilon^{-1}$, both assumed small. The problem is related to one that occurs in nonlinear mean-field dynamo theory (Malkus & Proctor 1975) and it is surmised that similar problems may be expected to appear in a variety of physical situations.

1. Introduction

The properties of thermal convection at low Prandtl number are of interest in the study of the convective regions of the sun and stars, where the effective conductivity of the stellar material is greatly enhanced by radiative processes. Prandtl numbers in the earth's liquid core are also thought to be rather less than unity, owing to the metallic nature of the core material. In these bodies the convection is characterized by large Reynolds numbers (inertial forces play a dominant role) except at very low amplitudes, and appears easily to become turbulent. Our understanding of turbulence is very limited, although Malkus (1954*a*, *b*), Busse (1969) and others have made progress towards an understanding of thermal turbulence by considering its onset as a series of discrete transitions and instabilities, leading eventually to time dependence and disorder. Other authors have chosen the opposite path: to ignore the instabilities (usually by suppressing all motions in a third space dimension) and to study the resulting laminar flow, both in order to elucidate the processes that drive the convection and limit its amplitude, and in order to obtain some idea of the macroscopic order that exists even in fully developed turbulence. Prominent among these are the numerical studies by Moore & Weiss (1973) and Jones *et al.* (1976), who solve the full equations for a layer heated from below (the Rayleigh–Bénard problem) by finite-difference and series-truncation methods. The first study is restricted to two-dimensional rolls and the second to flows with symmetry about a vertical axis. Comparison of these papers shows a striking contrast between the Prandtl number dependence of the evolved steady flow in the two geometries, especially near the critical Rayleigh number for the onset of convection. Figure 1, from Jones *et al.*, shows the convective heat transport (the Nusselt number N) as a function of the Prandtl number σ for different

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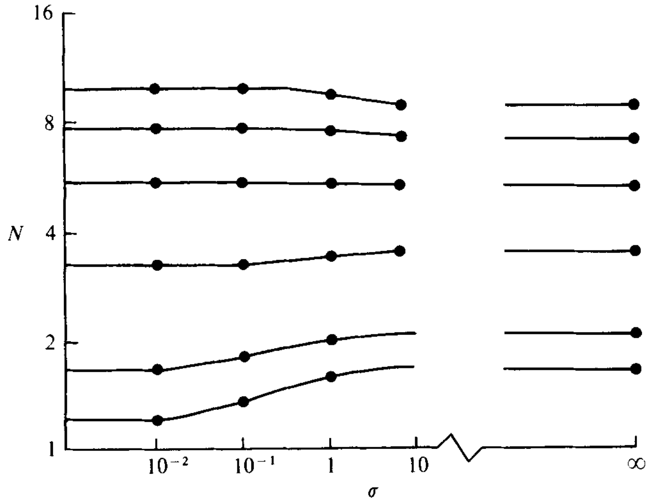

FIGURE 1. Nusselt number N as a function of Prandtl number σ for $R/R_0 = 1{\cdot}5$, 2, 6, 20, 50 and 100 in an upright cylinder (from Jones *et al.* 1976). Note the suppression of convection at small R as $\sigma \rightarrow 0$.

supercritical Rayleigh numbers R in the cylindrical geometry. It will be seen that at low R convection is inhibited significantly as σ becomes small: indeed, it is found that for $R < R_* \simeq 1{\cdot}32R_0$ (where R_0 is the critical Rayleigh number for the onset of convection) $N \rightarrow 1$ as $\sigma \rightarrow 0$, so that convection is suppressed entirely in this limit. For two-dimensional rolls on the other hand, the dependence of N on σ is very weak at all R.

Another interesting result of the Jones *et al.* study was the discovery that N was almost independent of σ at high enough R for both rolls and cylinders, in contrast to the theories of Spiegel (1971) and the numerical calculations of Gough, Spiegel & Toomre (1975), which predicted that $N = N(R\sigma)$ for $R \gg R_0$, $\sigma \ll 1$ in the cylindrical, but not the two-dimensional case. Further, the results of pivoting a perturbation scheme about the linear eigensolution corresponding to R_0 (e.g. Malkus & Veronis 1958), although satisfactory at all σ for the roll solutions, remain valid only for $(N-1) = O(\sigma^2)$ for the cylinders as $\sigma \rightarrow 0$.

The foregoing results suggested the possibility that for $\sigma \rightarrow 0$ there was some sort of asymptotic limit in which inertial constraints played a dominant role. The discrepancy between rolls and cylinders could then be explained by noting that for rolls the 'inertial equation' for the velocity $\mathbf{u}$,

$$\nabla \times (\mathbf{u}.\nabla\mathbf{u}) \equiv 0, \tag{1.1}$$

is identically satisfied by the linear eigensolution. For cylinders, (1.1) does not hold, so that adjustments due to the inertial terms (which appear dominantly in the scaling) occur at much lower amplitudes. Jones *et al.* demonstrated that a closed problem could be formulated in the limit $\sigma \rightarrow 0$ in which (1.1) was satisfied to leading order. This led naturally to the introduction of a 'second critical Rayleigh number' R_*, defined as the minimum value of R for which motions satisfying (1.1) can be driven by thermal forces. However, their problem as posed was highly nonlinear, and they were able to solve it, and demonstrate the existence of R_*, only by numerical techniques. In this paper, we study the same convection problem in a rather different

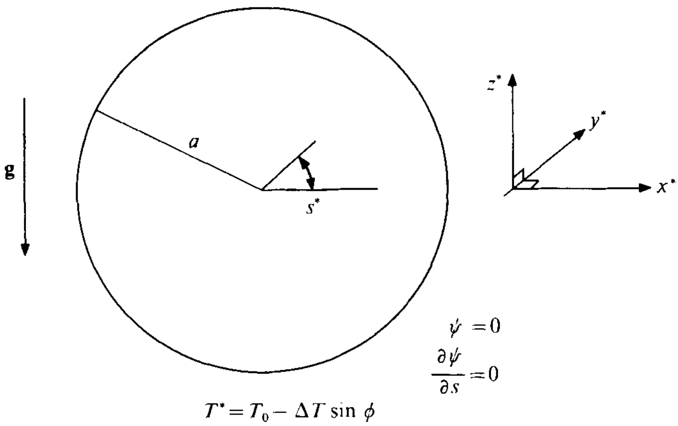

FIGURE 2. Geometry and boundary conditions for the horizontal cylinder problem.

geometry (motions in a *horizontal* circular cylinder), for which the equivalent nonlinear system is exactly solvable, so as to demonstrate the existence and certain variational properties of R_* analytically. We also indicate how the scheme can be extended to give solutions at small, but not zero, σ. In §2 the problem is formulated, and the existence of two possible asymptotic limits ($|\mathbf{u}|\to 0$, $\sigma|\mathbf{u}|^{-1}\to 0$) is noted. Solution by expansion methods is undertaken in §§3 and 4 and in §5 a conclusion evaluates the results and discusses the question of the stability, i.e. the observability, of the flow pattern near the transition point R_*. Some experiments of Rossby (1962, 1969) and Krishnamurti (1973) are invoked to suggest that there may be conditions on σ under which the transition point can be directly observed. The paper concludes with a discussion of other circumstances in which similar asymptotic limits may occur.

2. Formulation and isolation of the viscous and inviscid limits

2.1. *Geometry and equations*

We consider steady two-dimensional motions of a Boussinesq fluid in a horizontal cylinder $\mathscr{D}$ of radius $s^* = a$ with its axis along the y^* axis of a Cartesian co-ordinate system (figure 2). Here s^* and ϕ are polar co-ordinates, and gravity $\mathbf{g}$ is in the $-z^*$ direction. The cylinder wall is supposed fixed. The condition on the temperature T^* at the boundary is

$$T^* = T_0 - \Delta T \sin\phi, \tag{2.1}$$

corresponding to a perfectly conducting boundary. The velocity boundary conditions are the normal ones at a fixed surface. It should be noted that Jones *et al.* considered free boundaries only. We should have preferred to have done the same, but the combination of a circular cylinder (the only geometry for which the nonlinear problem posed below is solvable) and free boundaries produces effects that render the problem quite different from the normal Rayleigh–Bénard situation. In particular, motion can occur for any $R > 0$. The nature of the asymptotic solution in the limit $\sigma\to 0$ is unaffected by this change in the boundary conditions: that is, a solution of the problem can be found without any need for the introduction of boundary layers. For small σ, however, the deviations from the asymptotic solution involve viscous boundary layers, since the residual inertial forces cannot be balanced entirely by thermal forces near the boundaries. The effect on the relationship between R and $|\mathbf{u}|$ will be to introduce fractional powers of σ into the appropriate asymptotic expansion, as described in §4. The non-dimensionalized equations of steady motion under the Boussinesq approximation (in which the effects of density variation with temperature appear only as a buoyancy term, the fluid being otherwise considered incompressible) take the form

$$\sigma^{-1}[\mathbf{u}.\nabla\mathbf{u}]+\nabla p = R\theta\hat{\mathbf{z}}+\nabla^2\mathbf{u}, \qquad (2.2a)$$

$$\mathbf{u}.\nabla\theta = \nabla^2\theta, \qquad (2.2b)$$

$$\nabla.\mathbf{u} = 0, \quad \mathbf{u} = \nabla\times[\psi(s,\phi)\hat{\mathbf{y}}], \qquad (2.2c)$$

where the pressure p, velocity $\mathbf{u}$, temperature θ and radial distance s may be expressed in dimensional form (starred) as

$$p^* = (\nu/a^2)\,p, \quad \mathbf{u}^* = (\kappa/a)\,\mathbf{u}, \quad T^* = T_0+\theta\Delta T, \quad s^* = as \qquad (2.3)$$

(so that $\mathscr{D}$ has non-dimensional radius 1), where ν is the kinematic viscosity, κ the thermal conductivity and a the radius of $\mathscr{D}$. ($\hat{\mathbf{z}}$ is a unit vector in the z direction.) The two dimensionless parameters σ (Prandtl number) and R (Rayleigh number) take the form

$$\sigma = \nu/\kappa, \quad R = \alpha g\Delta T a^3/\kappa\nu, \qquad (2.4)$$

where α is the coefficient of thermal expansion and $g = |\mathbf{g}|$ is the acceleration due to gravity. If $R > 0$, the system is said to be unstably stratified. In this study, we suppose that $\sigma \ll 1$, so that the viscosity is 'small'. The boundary conditions on the stream function ψ and θ are

$$\psi = \partial\psi/\partial s = 0, \quad \theta = -\sin\phi \quad \text{for} \quad s = 1. \qquad (2.5)$$

We first note that (2.2)–(2.5) admit a basic hydrostatic state. If we set $\mathbf{u} \equiv 0$, we obtain for the temperature field θ_0

$$\left.\begin{aligned}\nabla^2\theta_0 &= 0,\\ \theta_0 &= -\sin\phi \quad \text{for} \quad s = 1,\end{aligned}\right\} \qquad (2.6)$$

and these have the unique solution

$$\theta_0 = -z. \qquad (2.7)$$

Substitution of (2.7) into (2.2a) shows that the thermal forces due to (2.7) can be balanced by a pressure gradient, and hence that (2.7) is the correct solution if $\mathbf{u} \equiv 0$. The problem then becomes one of the stability of this basic state.

2.2. *Expansion scheme for small field amplitudes*

Equations (2.2) are highly nonlinear, and in order to investigate them it is necessary to make approximations. We shall suppose that the convection (if it is present) is very weak, so that

$$\mathbf{u} = \epsilon\tilde{\mathbf{u}}, \quad \theta = -z+\epsilon\tilde{\theta} \quad \text{etc.}, \quad \epsilon \ll 1, \qquad (2.8)$$

where ϵ is a small dimensionless parameter independent of the scaling (but related to σ and R by the nonlinearity of the equations). Substituting into (2.2), we obtain

$$\eta^{-1}(\tilde{\mathbf{u}}.\nabla\tilde{\mathbf{u}})+\nabla\tilde{p} = R\tilde{\theta}\hat{\mathbf{z}}+\nabla^2\tilde{\mathbf{u}}, \qquad (2.9a)$$

$$\epsilon(\tilde{\mathbf{u}}.\nabla\tilde{\theta}) = \nabla^2\tilde{\theta}+\tilde{\mathbf{u}}.\hat{\mathbf{z}}, \qquad (2.9b)$$

where $\eta = \epsilon^{-1}\sigma$ is the reciprocal of the Reynolds number of the flow. All previous studies of the system (2.9) in other geometries have proceeded of necessity by supposing that both the nonlinear terms in (2.9) may be considered as small perturbations of a basic linear problem. This in turn implies that $\epsilon \ll \sigma$, which, if σ is very small, is a severe restriction on the accessible parameter space. To put it another way, the series obtained for R as a function of ϵ contains a power series in η^{-1}, so that its radius of convergence will be $O(\sigma)$. [The exceptions to this will occur when the offending inertial forces are irrotational: this turns out to be the case for two-dimensional rolls between free boundaries.] Although it does not seem possible to obtain analytical information unless $\epsilon \ll 1$, it is highly desirable to find some way of examining the region $1 \gg \epsilon \gg \sigma$. To do this, it is necessary to consider the inertial term $\eta^{-1}(\tilde{\mathbf{u}}\,.\,\nabla\tilde{\mathbf{u}})$ as the dominant term in any solution, and in what follows we show that this can be done as part of a consistent expansion scheme valid for large Reynolds numbers ($\eta \ll 1$).

We therefore define two limits as follows:

(*a*) The viscous limit: $1 \gg \eta^{-1} \gg \epsilon$.

(*b*) The inviscid limit: $\epsilon \ll 1, \quad \eta \ll 1$.

We discuss (*a*) in §3 and (*b*) in §4. The names of the limits are suggestive of the respective importance of viscous effects, and are taken from Malkus & Proctor (1975), where a rather similar asymptotic situation occurs in the nonlinear theory of mean-field dynamo models (the nonlinear effect there being due to Lorentz rather than inertial forces). It is helpful to note that (*a*) is a small and (*b*) a large Reynolds number expansion. Experience with other physical situations suggest that, while (*a*) is straightforward, (*b*) is likely to present difficulties and this is found to be the case here.

We are not aware of any previous use of limit (*b*) in any problem of thermal convection, except informally in the Jones *et al.* study.

In the next section, then, we derive the equations in the viscous limit and show that an expansion based on this limit cannot give any information about the region $\epsilon \geqslant \sigma$, in contrast to the horizontal roll solutions mentioned above, for which the viscous limit provides an excellent approximation even for $\epsilon > \sigma$. In §4, we show how the restrictions of this limit can be avoided and obtain solutions for much larger values of ϵ.

3. Solution in the viscous limit

3.1. *The eigenvalue problem*

In this section $1 \gg \eta^{-1} \gg \epsilon$, so that the nonlinear term in the temperature equation (2.9*b*) may be neglected. We now expand all quantities in powers of η^{-1}, so that

$$\left.\begin{aligned} R &= R + \eta^{-1}R_1 + \eta^{-2}R_2 + \dots, \\ \tilde{\mathbf{u}} &= \mathbf{u}_1 + \eta^{-1}\mathbf{u}_2 + \dots, \\ \tilde{\theta} &= \theta_1 + \eta^{-1}\theta_2 + \dots, \end{aligned}\right\} \quad \text{etc.,} \tag{3.1}$$

and fix η by requiring, for instance, that for some given† constant Q^2

$$\int \psi_1 \frac{\partial \theta_1}{\partial x}\, dV = Q^2 = \int \bar{\psi} \frac{\partial \theta_1}{\partial x}\, dV$$

† This constant is determined by the normalization of §4.

(the volume integrals being understood to be over a unit length of $\mathscr{D}$). Then we obtain, to leading order in η^{-1},

$$\nabla p_1 = R_0 \theta_1 \hat{\mathbf{z}} + \nabla^2 \mathbf{u}_1, \tag{3.2a}$$

$$0 = \nabla^2 \theta_1 + \mathbf{u}_1 . \hat{\mathbf{z}}, \tag{3.2b}$$

and these, together with the relevant boundary conditions from (2.5), constitute a linear eigenvalue problem for R_0. The inertial term can be treated as a perturbation in the viscous limit. This linear problem has been studied by many authors in varied geometries, starting with Rayleigh (1916); Chandrasekhar (1961, chap. 2) gives a particularly full account. The eigenvalue problem in the particular geometry discussed here has been treated by Weinbaum (1964). He obtains $R_0 = 567$ as the critical eigenvalue, but this value is incorrect owing to his use of an erroneous variational principle. Unfortunately, this geometry does not permit a separable solution. Examination of the equation reveals that the lowest eigenmode must be of the form

$$\left.\begin{aligned} \psi_1(s,\phi) &= \psi_{10}(s) + \psi_{12}(s)\cos 2\phi + \psi_{14}(s)\cos 4\phi + \ldots, \\ \theta_1(s,\phi) &= \theta_{11}(s)\cos\phi + \theta_{13}(s)\cos 3\phi + \theta_{15}(s)\cos 5\phi + \ldots . \end{aligned}\right\} \tag{3.3}$$

It is not our object to give details of this limit, which can in any case be found elsewhere for other geometries; for example Jones *et al.* give details of the method for their vertical cylinder; Malkus & Veronis (1958) were the first to treat nonlinear Rayleigh–Bénard convection in a plane layer; a more refined account is given by Schlüter, Lortz & Busse (1965). We shall seek only certain general features of the solution, in particular an upper bound for R_0. It can be shown that (3.2) possesses a variational principle: R_0 is given by the extremum† of the functional

$$R[\psi,\theta] = \frac{\int |\nabla^2\psi|^2 dV \times \int |\nabla\theta|^2 dV}{\left(\int \theta(\partial\psi/\partial x)\, dV\right)^2}, \tag{3.4}$$

where ψ and θ are sufficiently regular functions satisfying

$$\psi = \partial\psi/\partial s = \theta = 0 \quad \text{at} \quad s = 1. \tag{3.5}$$

Without solving (3.2), therefore, an upper bound for R_0 may be found by choosing suitable trial functions ψ and θ. Truncating the expansion of ψ at the $\cos m\phi$ term and that of θ_1 at the $\cos n\phi$ term yields variational equations which may be solved exactly (in terms of Bessel functions) to yield

$$R = 408{\cdot}0 \quad \text{with} \quad m = 2, \quad n = 1, \quad R = 406{\cdot}7 \quad \text{with} \quad m = 4, \quad n = 3. \tag{3.6}$$

It is clear that the eigenvalue R_0 is very close to 406. Equation (3.6) is sufficient to show that R_0 differs from the R_* of the next section (one of the main results of this study).

† This extremum can be shown to be a minimum if the eigenfunctions of (3.2) are supposed complete (see Chandrasekhar 1961, p. 30).

3.2. *Extension to finite amplitude*

If we suppose that ψ_1 and θ_1 have been found, and that† $\epsilon \ll \eta^{-3}$, we may in principle determine R_1, R_2, etc., by considering higher-order terms in the η^{-1} expansion. At $O(\eta^{-2})$ and $O(\eta^{-3})$ we have, respectively,

$$-R_1\theta_1\hat{\mathbf{z}}+\mathbf{u}_1.\nabla\mathbf{u}_1 = -\nabla p_2+R_0\theta_2\hat{\mathbf{z}}+\nabla^2\mathbf{u}_2, \tag{3.7a}$$

$$0 = \nabla^2\theta_2+\mathbf{u}_2.\hat{\mathbf{z}} \tag{3.7b}$$

and

$$-R_1\theta_2\hat{\mathbf{z}}-R_2\theta_1\hat{\mathbf{z}}+\mathbf{u}_1.\nabla\mathbf{u}_2+\mathbf{u}_2.\nabla\mathbf{u}_1 = -\nabla p_3+R_0\theta_3\hat{\mathbf{z}}+\nabla^2\mathbf{u}_3, \tag{3.8a}$$

$$0 = \nabla^2\theta_3+\mathbf{u}_3.\hat{\mathbf{z}}. \tag{3.8b}$$

Clearly, these equations are inhomogeneous versions of (3.2) and it is well known from the theory of similar systems (see, for example, Schlüter *et al.* 1965) that steady solutions exist only if the inhomogeneous left-hand sides are orthogonal to the adjoint of the original eigensolution. Since, in this case, the eigenvalue problem is self-adjoint (as may easily be verified), these solvability conditions involve the eigensolution $(\mathbf{u}_1, \theta_1)$ itself, and take the form

$$R_1\int\theta_1\mathbf{u}_1.\hat{\mathbf{z}}\,dV = \int(\mathbf{u}_1.\nabla\mathbf{u}_1).\mathbf{u}_2\,dV = 0, \tag{3.9a}$$

$$R_2\int\theta_1\mathbf{u}_1.\hat{\mathbf{z}}\,dV+R_1\int\theta_2\mathbf{u}_1.\hat{\mathbf{z}}\,dV = \int(\mathbf{u}_2.\nabla\mathbf{u}_1+\mathbf{u}_1.\nabla\mathbf{u}_2).\mathbf{u}_1\,dV. \tag{3.9b}$$

Thus after some manipulation, and use of (3.7*b*) and (3.8*b*), we obtain

$$\begin{aligned} Q^2R_2 &= [\textstyle\int|\nabla^2\psi_2|^2dV+R_0\int|\nabla\theta_2|^2dV-2R_0\int\theta_2\mathbf{u}_2.\hat{\mathbf{z}}\,dV] \\ &\geqslant 2R_0^{\frac{1}{2}}[(\textstyle\int|\nabla^2\psi_2|^2dV\int|\nabla\theta_2|^2dV)^{\frac{1}{2}}-R_0^{\frac{1}{2}}\int\theta_2\mathbf{u}_2.\hat{\mathbf{z}}\,dV] \end{aligned} \tag{3.10}$$

and $R_2 \geqslant 0$ from the minimality property of R_0: equality holds only if ψ_2, $\theta_2 = 0$ or if they are solutions of the homogeneous equations (2.9). However, these functions actually satisfy (3.7), and since $R_1 = 0$ this equation is homogeneous if and only if

$$\nabla\times(\mathbf{u}_1.\nabla\mathbf{u}_1) \equiv 0. \tag{3.11}$$

It may be shown [equation (4.14)] that this cannot hold since $\mathbf{u}_1$ depends on ϕ and satisfies the full viscous boundary conditions. We therefore have

$$R = R_0+\epsilon^2\sigma^{-2}R_2+\dots. \tag{3.12}$$

We may extend the expansion scheme to include corrections to R of order $\epsilon\eta^{-1}$ and order ϵ^2, there being no corrections $O(\epsilon)$ since reversing the direction of the flow does not affect the physics of the problem. Then the leading-order terms in the expansion of $R(\epsilon)$ can be shown by similar methods to be

$$R = R_0+\epsilon^2\sigma^{-2}R_2+O(\epsilon^4\sigma^{-4})+\epsilon^2\sigma^{-1}A_1+\epsilon^2A_2, \tag{3.13}$$

where A_1 and A_2 are positive numbers of order unity, so that (3.12) is correct to leading order in η^{-1} provided only that $\sigma \ll 1$. If the expansion of R were continued further, it is clear that part of the expansion is a power series in η^{-1}, which is presumably valid only if $\eta^{-1} = O(1)$ or less. Certainly, the slope of the R, ϵ^2 curve depends strongly on σ for these small amplitudes, in contrast to rolls with free plane boundaries, but like the

† This assumption is made here for simplicity; we shall see, however, that the results of this subsection do not depend on such a restriction.

problem of Jones *et al.* In the next section, we show that this behaviour does not persist when $\epsilon \gg \sigma$ and that solutions at these larger amplitudes are independent of σ to leading order.

4. Solution in the inviscid limit ($1 \gg \epsilon \gg \sigma$)

4.1. *Formulation*

The basis for the limit discussed in this section is contained in the remark that if $\epsilon \ll 1$ but $\sigma \ll \epsilon$ then the Reynolds number

$$\eta^{-1} \equiv \epsilon\sigma^{-1} \tag{4.1}$$

is very large, and η is therefore a small parameter in which it might be appropriate to make an expansion. We therefore suppose $\epsilon \ll 1$ and $\eta \ll 1$ and expand all quantities in powers of ϵ and η:

$$\epsilon\tilde{\mathbf{u}} = \sum_{i=1}^{\infty} \epsilon^i \mathbf{u}_{i0} + \eta \sum_{i=1}^{\infty} \epsilon^i \mathbf{u}_{i1} + O(\eta^2\epsilon)$$

and similarly for $\tilde{\theta}$, $\tilde{p}$, etc., with

$$R = R_{00} + \epsilon R_{10} + \ldots + \eta R_{01} + \ldots . \tag{4.2}$$

Then on substituting into (2.9), it is clear that, to leading order in η and ϵ, we have

$$\nabla \times (\mathbf{u}_{10} . \nabla \mathbf{u}_{10}) = 0. \tag{4.3}$$

Equation (4.3) states that, in the limit of infinite Reynolds number, there is nothing but pressure to balance the inertial forces, which must therefore be irrotational. Equation (4.3) is nonlinear, and so the solution we produce below is nonlinear at the outset, instead of being only a small perturbation from a linear result. In terms of the stream function ψ_{10} and vorticity ω_{10}, (4.3) can be written as

$$\omega_{10} = -\nabla^2\psi_{10} = f(\psi_{10}), \tag{4.4}$$

where f is arbitrary. Equation (4.4) is to be solved subject to

$$\psi_{10} = \partial\psi_{10}/\partial s = 0 \quad \text{at} \quad s = 1, \tag{4.5}$$

so that the boundary conditions would seem to overdetermine the problem, (4.4) being of second order. It will be shown below, however, that (4.4) and (4.5) can be solved in this geometry without the need to invoke viscous boundary layers at leading order.

For the moment, we suppose that (4.4) can be solved for any given f. The solution is still highly degenerate, and we need some constraint which will determine f and hence ψ_{10}. Batchelor (1956) has noted that, if a flow field consists (in part at least) of closed streamlines, then the inertial and pressure forces make no contribution to the energy budget of each individual closed streamline, which is accordingly determined (in the case in hand) by a balance between thermal driving and viscous dissipation. Specifically, if we rewrite (2.2*a*) as

$$\mathbf{I} = \sigma^{-1}[\boldsymbol{\omega} \times \mathbf{u} + \nabla p'] = R\theta\hat{\mathbf{z}} + \nabla^2\mathbf{u} \tag{4.6}$$

and then examine

$$\oint_c \mathbf{I} . \mathbf{dl},$$

where **dl** is parallel to **u** and the integral is taken around any closed streamline c, it is easy to see that

$$\oint_c \mathbf{I}.\mathbf{dl} = 0 \tag{4.7}$$

and hence that

$$R\oint_c \theta\hat{\mathbf{z}}.\mathbf{dl} + \oint_c \nabla^2\mathbf{u}.\mathbf{dl} = 0 \tag{4.8}$$

for *each* streamline. (Note that σ is absent from this *exact* relation.) Thus the undetermined residual inertial forces

$$\mathbf{u}_{11}.\nabla\mathbf{u}_{10} + \mathbf{u}_{10}.\nabla\mathbf{u}_{11} + O(\eta),$$

which appear in the ordering at the same level as the buoyancy and viscous terms, are eliminated in the integral. If we expand (4.8) in powers of ϵ, we obtain at leading order

$$R_{00}\oint_{c_{10}} \theta_{10}\hat{\mathbf{z}}.\mathbf{dl} + \oint_{c_{10}} \nabla^2\mathbf{u}_{10}.\mathbf{dl} = 0, \tag{4.9}$$

where c_{10} is a streamline of ψ_{10}. We determine θ_{10} from the leading-order terms in (2.2*b*), which give [cf. (3.2*b*)]

$$\nabla^2\theta_{10} + \mathbf{u}_{10}.\hat{\mathbf{z}} = 0. \tag{4.10}$$

Equations (4.4), (4.9) and (4.10) form a closed problem which, if well posed and solvable, will yield f, ψ_{10}, θ_{10} and R_{00}. An exactly analogous problem was obtained by Jones *et al.* but they did not attempt to solve it directly: the problem is of a very difficult implicit type, since the domain of integration for (4.9) depends on ψ_{10}. It is possible that some iterative method will yield results in the general case. In the present geometry, we shall show that the equations take the form of a linear fourth-order boundary-value problem of classic type, which can be solved to yield a value for R_{00}.

4.2. *Solution to leading order and determination of R_{00}*

It is clear by inspection that a family of solutions of (4.4) and (4.5) is

$$\omega_{10} = \omega_{10}(s); \quad \psi_{10} = \psi_{10}(s); \quad \psi_{10} = \partial\psi_{10}/\partial s = 0 \quad \text{at} \quad s = 1. \tag{4.11}$$

We may prove quite straightforwardly that (4.11) gives the only solutions that are compatible with the fourth-order boundary conditions (4.5). If we write (4.4) as

$$\frac{\partial^2\psi_{10}}{\partial s^2} + \frac{1}{s}\frac{\partial\psi_{10}}{\partial s} + \frac{1}{s^2}\frac{\partial^2\psi_{10}}{\partial\phi^2} = -f(\psi_{10}) \tag{4.12}$$

then taking the ϕ derivative and setting $s = 1$ yields

$$\frac{\partial^2}{\partial s^2}\left(\frac{\partial\psi_{10}}{\partial\phi}\right)\bigg|_{s=1} = -\frac{\partial^2\psi_{10}}{\partial s\,\partial\phi}\bigg|_{s=1} - \frac{\partial^3\psi_{10}}{\partial\phi^3}\bigg|_{s=1} - \frac{\partial f}{\partial\psi}\frac{\partial\psi_{10}}{\partial\phi}\bigg|_{s=1} \tag{4.13}$$

$$= 0, \quad \text{since} \quad \frac{\partial\psi_{10}}{\partial\phi} = \frac{\partial\psi_{10}}{\partial s} = 0 \quad \text{at} \quad s = 1.$$

Repeated differentiation of (4.12) then yields

$$\frac{\partial^n}{\partial s^n}\left(\frac{\partial\psi_{10}}{\partial\phi}\right)\bigg|_{s=1} = 0 \quad \text{for all} \quad n \geqslant 0. \tag{4.14}$$

If it is supposed that ψ_{10} can be expanded as a Taylor series about $s = 1$, as seems reasonable, it follows that $\partial\psi_{10}/\partial\phi = 0$, and hence that (4.11) gives the only solutions.†

We are now able to make a remarkable simplification. Equation (4.11) implies that the leading-order streamlines are all circles, *independently of the form of f*, and hence that the streamline integral can be evaluated before f is known. For a given streamline, $\mathbf{dl} \propto \hat{\boldsymbol{\phi}}\, d\phi$, where $\hat{\boldsymbol{\phi}}$ is a unit vector in the ϕ direction, and (4.9) then becomes

$$R_{00}\int_0^{2\pi} \theta_{10}\cos\phi\, d\phi + 2\pi(\nabla^2\mathbf{u}_{10})_\phi = 0, \tag{4.15}$$

since $\mathbf{u}_{10}$ is in the ϕ direction. If $\mathbf{u} = v_{10}(s)\,\hat{\boldsymbol{\phi}}$ $(v_{10} = -\partial\psi_{10}/\partial s)$, then (4.10) can be written as

$$\nabla^2\theta_{10} = -v_{10}\cos\phi \tag{4.16}$$

and hence if $\theta_{10} = \hat{\theta}_{10}\cos\phi$ we obtain

$$D^2\hat{\theta}_{10} = -v_{10}, \tag{4.17}$$

where

$$D^2 = \frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} - \frac{1}{s^2}.$$

Equation (4.15) can now be evaluated to yield

$$\tfrac{1}{2}R_{00}\hat{\theta}_{10} + D^2 v_{10} = 0, \tag{4.18}$$

where the $\frac{1}{2}$ comes from the integration of $\cos^2\phi$ round the azimuth. The fourth-order system (4.17) and (4.18), subject to the boundary conditions

$$\hat{\theta}_{10} = v_{10} = 0 \quad \text{on} \quad s = 1, \tag{4.19}$$

constitutes an eigenvalue problem for R_{00}. This system is independent of σ, and therefore represents an asymptotic limit as $\sigma \to 0$. The equations are easily solved to yield

$$\left.\begin{aligned} \hat{\theta}_{10} = KJ_1(\alpha s), \quad v_{10} = \alpha^2 K J_1(\alpha s), \\ J_1(\alpha) = 0, \quad \alpha \triangleq 3{\cdot}82\ldots, \\ R_{00} = 2\alpha^4 \triangleq 432, \quad K^2 = 2/J_0^2(\alpha)\,\alpha^2, \end{aligned}\right\} \tag{4.20}$$

where $J_1(x)$ is a Bessel function of order 1 and K is a normalization constant‡ chosen such that

$$\int_0^1 \hat{\theta}_{10} v_{10} s\, ds = 1.$$

R_{00} is the 'second critical eigenvalue' R_* already alluded to.

This system, then, is much simpler both in structure and in solution than the full linear eigenvalue problem. At leading order, the fluid is constrained to move in circles by the inertial forces. Since this is not the optimum mode for turning thermal into

† It is possible that there are other solutions to the problem in which the inertial forces are irrotational away from the boundary, but can match the boundary conditions only by means of thin viscous layers. While we have not investigated any of these possible solutions, we are sure that (involving additional viscous loss as they do) they would correspond to a critical Rayleigh number scaled with σ^{-1} in some way: that is, very much greater than R_{00}. See the conclusion of this paper.

‡ This fixes the previously undetermined constant Q^2 defined after (3.1).

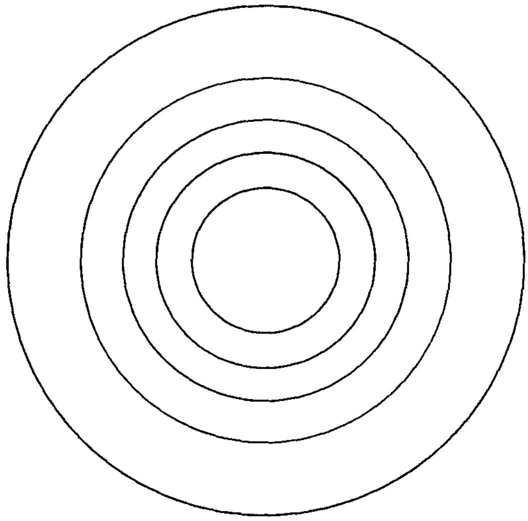

FIGURE 3. Streamlines of ψ_{10}. Assuming that $\psi_{10}(0) = 1$, the lines are equal intervals of 0·2 in ψ_{10}.

kinetic energy, the unstable temperature gradient needed will be greater than that given by the R_0 of §3. Of course, for a given σ, η is small only if ϵ is bounded away from zero. However, if σ is sufficiently small, then if $\epsilon \gg \sigma$ the R, ϵ^2 curve will appear, when extrapolated back to $\epsilon = 0$, to pass through R_{00}. Hence R_{00} can appear to be a 'critical Rayleigh number' analogous to R_0. Figure 3 shows the streamlines of this eigensolution.

Having obtained the zero-order solution, we may use it as a basis for expansion schemes in both ϵ and η. We treat the ϵ expansion in some detail below; the η expansion, although involving equations that cannot be solved analytically, does yield the first-order correction to R without resort to numerical techniques.

Before passing on to this programme, we should note that in this case R_{00} can be obtained from a variational principle: R_{00} is the minimum of the functional

$$R[\psi, \theta] = \frac{\int |\nabla^2\psi|^2 \, dV \times \int |\nabla\theta|^2 \, dV}{\left(\int \theta(\partial\psi/\partial x) \, dV\right)^2} \tag{4.21}$$

[cf. (3.4)] if ψ is restricted to being a function of s alone. A similar conjecture for the Jones *et al.* problem has now been disproved (personal communication from N. O. Weiss).

4.3. *The ϵ expansion*

If we now suppose that $\eta \ll \epsilon^2$ (this becomes more and more easy to satisfy as ϵ^2 becomes larger), we may continue the ϵ expansion while still neglecting all terms in η except those of zero order; that is, we extend the $\eta \to 0$ limit to finite amplitude. It is clear by comparison with (4.3) that the inertial forces must be irrotational at all orders up to $O(\epsilon^4)$; therefore we have

$$\psi_{20} = \psi_{20}(s), \quad \psi_{30} = \psi_{30}(s), \quad \text{etc.} \tag{4.22}$$

We may determine these functions by making use of the higher-order equations in the expansion of (4.8) in powers of ϵ [note that the streamlines remain circular, by virtue of (4.22)]. These give

$$\left.\begin{aligned} &R_{10}\int_0^{2\pi}\theta_{10}\hat{\mathbf{z}}.\hat{\boldsymbol{\phi}}\,d\phi+R_{00}\int_0^{2\pi}\theta_{20}\hat{\mathbf{z}}.\hat{\boldsymbol{\phi}}\,d\phi+\int_0^{2\pi}\nabla^2\mathbf{u}_{20}.\hat{\boldsymbol{\phi}}\,d\phi=0,\\ &R_{20}\int_0^{2\pi}\theta_{10}\hat{\mathbf{z}}.\hat{\boldsymbol{\phi}}\,d\phi+R_{10}\int_0^{2\pi}\theta_{20}\hat{\mathbf{z}}.\hat{\boldsymbol{\phi}}\,d\phi+R_{00}\int_0^{2\pi}\theta_{30}\hat{\mathbf{z}}.\hat{\boldsymbol{\phi}}\,d\phi+\int_0^{2\pi}\nabla^2\mathbf{u}_{30}.\hat{\boldsymbol{\phi}}\,d\phi=0\end{aligned}\right\}\tag{4.23}$$

and we also have the thermal equations

$$\left.\begin{aligned}\nabla^2\theta_{20}&=-\mathbf{u}_{20}.\hat{\mathbf{z}}+\mathbf{u}_{10}.\nabla\theta_{10},\\ \nabla^2\theta_{30}&=-\mathbf{u}_{30}.\hat{\mathbf{z}}+\mathbf{u}_{20}.\nabla\theta_{10}+\mathbf{u}_{10}.\nabla\theta_{20}\end{aligned}\right\}\tag{4.24}$$

from (2.2*b*). Note that all the nonlinearity appears in the thermal equations.

We may simplify (4.23) and (4.24) by extracting the ϕ dependence of the temperature variables. We write

$$\left.\begin{aligned}\theta_{20}&=\tilde{\theta}_{20}(s)\sin\phi+\hat{\theta}_{20}(s)\cos\phi,\\ \theta_{30}&=\tilde{\theta}_{30}(s)\sin\phi+\hat{\theta}_{30}(s)\cos\phi,\\ \mathbf{u}_{20}&=v_{20}(s)\,\hat{\boldsymbol{\phi}},\quad \mathbf{u}_{30}=v_{30}(s)\,\hat{\boldsymbol{\phi}}.\end{aligned}\right\}\tag{4.25}$$

(This can be done because all the nonlinear terms can consistently be taken as proportional to $\sin\phi$ or $\cos\phi$.) Equations (4.23) then become

$$\tfrac{1}{2}R_{10}\hat{\theta}_{10}+\tfrac{1}{2}R_{00}\hat{\theta}_{20}+D^2v_{20}=0, \tag{4.26a}$$

$$\tfrac{1}{2}R_{20}\hat{\theta}_{10}+\tfrac{1}{2}R_{10}\hat{\theta}_{20}+\tfrac{1}{2}R_{00}\hat{\theta}_{30}+D^2v_{30}=0 \tag{4.26b}$$

and (4.24) can be written as

$$D^2\tilde{\theta}_{20}=-(v_{10}/s)\,\hat{\theta}_{10},\qquad D^2\hat{\theta}_{20}=-v_{20}, \tag{4.27a,b}$$

$$D^2\tilde{\theta}_{30}=-(v_{10}/s)\,\hat{\theta}_{20}-(v_{20}/s)\,\hat{\theta}_{10}, \tag{4.27c}$$

$$D^2\hat{\theta}_{30}=-v_{30}+(v_{10}/s)\,\tilde{\theta}_{20}. \tag{4.27d}$$

It is now an easy matter to verify that (4.26*a*) and (4.27*b*) can be solved if and only if $R_{10}=0$ (we should expect this result in any case, since there is no reason for the physics of the problem to depend on the *sign* of ϵ). In this case, $\hat{\theta}_{20}$ and v_{20} are proportional to $\hat{\theta}_{10}$ amd v_{10} respectively, and they may be taken as zero by using the normalization condition in §3.1. Clearly $\tilde{\theta}_{30}$ is also zero, and we are left with the three equations

$$\left.\begin{aligned}D^2\tilde{\theta}_{20}&=-(K^2\alpha^2/s)\,J_1^2(\alpha s),\\ D^2\hat{\theta}_{30}&=-v_{30}+(\alpha^2/s)\,J_1(\alpha s)\,\tilde{\theta}_{20},\\ \tfrac{1}{2}R_{20}\hat{\theta}_{10}+\tfrac{1}{2}R_{00}\hat{\theta}_{30}&+D^2v_{30}=0,\end{aligned}\right\}\tag{4.28}$$

with $\tilde{\theta}_{20}=\hat{\theta}_{30}=v_{30}=0$ at $s=1$. We now fix R_{20} by the requirement that (4.28) should possess a solution, as for the viscous limit above. After some manipulation and use of the equations satisfied by v_{10} and $\hat{\theta}_{10}$ we find

$$\frac{R_{20}}{2}\int_0^1\hat{\theta}_{10}v_{10}s\,ds=-\int_0^1 v_{10}D^2\left(\frac{\alpha^2}{s}J_1(\alpha s)\,\tilde{\theta}_{20}\right)s\,ds \tag{4.29}$$

and hence

$$R_{20} = -2\alpha^2 \int_0^1 \tilde{\theta}_{20} D^2 \tilde{\theta}_{20} s\, ds > 0.$$

It is therefore only necessary to find $\tilde{\theta}_{20}$ in order to find R_{20}. The solution of (4.27*a*) is

$$\tilde{\theta}_{20} = \tfrac{1}{2}\alpha^2 K^2 [s[J_0^2(\alpha s) - J_0^2(\alpha)] + sJ_1^2(\alpha s) - \alpha^{-1} J_0(\alpha s) J_1(\alpha s)] \tag{4.30}$$

and hence

$$R_{20} = \alpha^6 K^4 \left[\int_0^1 sJ_1^2(\alpha s)\,[J_0^2(\alpha s) - J_0^2(\alpha)]\, ds + \int_0^1 sJ_1^4(\alpha s)\, ds - \frac{1}{\alpha}\int_0^1 J_0(\alpha s) J_1^3(\alpha s)\, ds \right]. \tag{4.31}$$

Hence the initial slope of the ϵ^2, R line is positive and $O(1)$. Clearly this perturbation method can be extended to all orders in ϵ. Indeed, a full numerical solution would not be difficult since there are only three independent equations in one space variable to be solved.

4.4. *The η expansion*

We now suppose that $\eta \gg \epsilon^2$, so that we may consider the $O(\eta)$ equations while neglecting corrections of order ϵ^2 or higher. Then at $O(\eta)$, (2.9) becomes

$$\begin{aligned} &\mathbf{u}_{11} \cdot \nabla \mathbf{u}_{10} + \mathbf{u}_{10} \cdot \nabla \mathbf{u}_{11} + \epsilon(\mathbf{u}_{21} \cdot \nabla \mathbf{u}_{10} + \mathbf{u}_{10} \cdot \nabla \mathbf{u}_{21} + \mathbf{u}_{20} \cdot \nabla \mathbf{u}_{11} + \mathbf{u}_{11} \cdot \nabla \mathbf{u}_{20}) \\ &\quad + \nabla(p_{10} + \epsilon p_{20}) = R_{00}\theta_{10}\hat{\mathbf{z}} + \nabla^2 \mathbf{u}_{10} + \epsilon[R_{00}\theta_{20}\hat{\mathbf{z}} + \nabla^2 \mathbf{u}_{20}] + [\eta \nabla^2 \mathbf{u}_{11}] + O(\eta), \quad (4.32a) \\ &\qquad \nabla^2 \theta_{11} + \mathbf{u}_{11} \cdot \hat{\mathbf{z}} = O(\epsilon). \quad (4.32b) \end{aligned}$$

If $\epsilon \ll \eta$, all the terms $O(\epsilon)$ may be neglected. If $\epsilon \gg \eta$, $\mathbf{u}_{20}$ ($= 0$) and θ_{20} are given by the analysis of the previous subsection up to $O(\epsilon^2)$. Then (4.32*a*) is linear in $\mathbf{u}_{11} + \epsilon \mathbf{u}_{21}$ and we may consider separately the effect of the forcing term $\epsilon R_{00}\theta_{20}\hat{\mathbf{z}}$ that gives rise to $\mathbf{u}_{21}$. It can easily be shown that this leads to a correction to R of order $\epsilon\eta$. As shown below, the correction to R due to $\mathbf{u}_{11}$ is $O(\eta^{\frac{4}{3}} \ln \eta)$, so for the validity of what follows we must require that $\epsilon \ll |\eta^{\frac{1}{3}} \ln \eta|$, which is guaranteed since $\eta \gg \epsilon^2$. We therefore neglect henceforward all terms $O(\epsilon)$ in (4.32).

As well as (4.32) we use the power integral

$$R \int \theta \frac{\partial \psi}{\partial x}\, dV - \int (\nabla^2 \psi)^2\, dV = 0 \tag{4.33}$$

obtained by integrating (4.8) over all streamlines. Expanding to $O(\eta)$, we obtain

$$\eta R_{01} \int |\nabla \theta_{10}|^2\, dV + 2R_{00}\eta \int \theta_{10} \frac{\partial \psi_{11}}{\partial x}\, dV - 2\eta \int (\nabla^2 \psi_{10} \cdot \nabla^2 \psi_{11})\, dV = 0 \tag{4.34}$$

(again, (2.9*b*) has been used), where ηR_{01} is the change in R to leading order. It will emerge that it is not necessary to find the full leading-order solution to determine R_{01}.

Equation (4.32*a*) may be simplified by taking the y component of the curl, which yields

$$\mathbf{u}_{11} \cdot \nabla \omega_{10} + \mathbf{u}_{10} \cdot \nabla \omega_{11} = \tfrac{1}{2} R_{00} K \alpha J_2(\alpha s) \cos 2\phi + [\eta \nabla^2 \omega_{11}] \tag{4.35}$$

since that part of $R_{00}\, \partial\theta_{10}/\partial x$ independent of ϕ is balanced by $\nabla^2 \omega_{10}$ (this constitutes the basic stability problem for $\eta \to 0$). Then, if we write

$$\mathbf{u}_{11} = \mathrm{Im}\, [\nabla \times (\chi_{11}(s)\, e^{2i\phi}\, \hat{\mathbf{y}})] \tag{4.36}$$

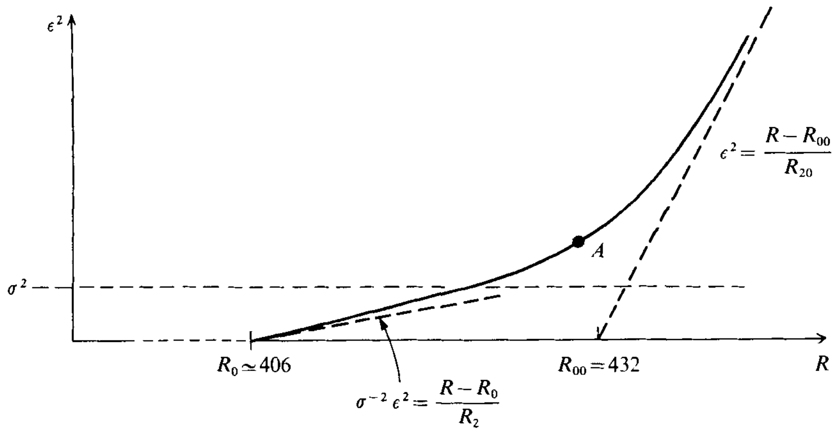

FIGURE 4. Sketch of the relation between R and ϵ^2 near the inviscid limit (η and ϵ small). As $\sigma \to 0$ the slope near R_0 tends to zero like σ^2; hence A, the point of minimum distance from R_{00}, tends to R_{00} as $\sigma \to 0$; points to the left of A go to the R axis while points to the right go to the line $\epsilon^2 = (R-R_{00})/R_{20}$.

we obtain

$$2K\alpha^2 s^{-1} J_1(\alpha s)\,[L^2\chi_{11} + \alpha^2\chi_{11}] = -\tfrac{1}{2}R_{00}K\alpha J_2(\alpha s) - [i\eta L^4\chi_{11}], \tag{4.37}$$

where

$$L^2 \equiv \frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} - \frac{4}{s^2}.$$

$\chi_{11}(s)$ must satisfy the boundary conditions

$$\chi_{11} = \partial\chi_{11}/\partial s = 0 \quad \text{at} \quad s = 0, 1. \tag{4.38}$$

It is immediately clear that, although there are certainly solutions that satisfy the conditions at $s = 0$, there is no possibility of the reduced system (with the viscous term neglected) being well behaved near $s = 1$ since $J_1(\alpha) = 0$. In order to satisfy the conditions there, it is necessary to match the solution of the reduced system (valid away from $s = 1$) to an inner solution near the boundary where the viscous term has to be taken into account. Details of the matching process are given in the appendix. The main result derived therein is that to leading order in the interior

$$\chi_{11} = P(s) + C_\eta \eta^{\frac{1}{3}} \ln(\eta^{\frac{1}{3}})\, J_2(\alpha s), \tag{4.39}$$

where $P(s)$ is a real function of s and C_η is a complex constant calculated as part of the matching process. The logarithm appears since the derivative of the outer solution $P(s)$ is logarithmically singular near $s = 1$. From (A 11) we have

$$C_\eta = C_\delta/(2K\alpha^3 J_2(\alpha))^{\frac{1}{3}} \tag{4.40}$$

plus terms of order $1/\ln \eta^{\frac{1}{3}}$. It is now clear that the last term in (4.34) vanishes to leading order, since ψ_{10} is independent of ϕ and $\psi_{11} \propto e^{2i\phi}$. Now $\theta_{10} \propto \cos\phi$ and $\partial\psi_{11}/\partial x$ contains terms proportional to $\sin\phi$, $\cos\phi$, $\sin 3\phi$ and $\cos 3\phi$. Thus the only part that contributes to (4.34)† is

$$\frac{1}{2}\left(\frac{\partial}{\partial s} + \frac{2}{s}\right) \operatorname{Im}(\chi_{11}) \cos\phi = \tfrac{1}{2}\alpha \operatorname{Im}(C_\eta)\, \eta^{\frac{1}{3}} \ln \eta^{\frac{1}{3}}\, J_1(\alpha s) \cos\phi. \tag{4.41}$$

† It may be verified that there is no contribution, at this order, from the viscous boundary layer.

Hence $R_{01} = \eta^{\frac{1}{3}} \ln \eta^{\frac{1}{3}} B$, where

$$B = -\mathrm{Im}\,(C_\eta)\, R_{00}/\alpha K \simeq 650. \tag{4.42}$$

Since $\ln \eta^{\frac{1}{3}} < 0$ for small η, $R_{01} < 0$: this might be expected in any case, since it is plausible that $R(\epsilon)$ would be a monotonic function. We have now found the leading-order corrections to R when $\eta \ll \epsilon^2$ and when $\eta \gg \epsilon^2$. Since corrections to R from the $O(\epsilon\eta)$ equation are much smaller than those from the $O(\eta)$ equations and since there are no corrections $O(\epsilon)$ to R, we may assume that

$$|\eta^{\frac{1}{3}} \ln \eta^{\frac{1}{3}}| \gg \epsilon \gg |\eta^{\frac{4}{3}} \ln \eta^{\frac{1}{3}}| = O(\epsilon^2). \tag{4.43}$$

Then to leading order the two corrections we have found can be combined to yield

$$R = R_{00} + \epsilon^2 R_{20} + \tfrac{1}{3}\sigma^{\frac{4}{3}} \ln(\sigma)\, B\epsilon^{-\frac{4}{3}} + \text{smaller terms} \tag{4.44}$$

and this is sketched in figure 4. It is clear from this expression that as $\sigma \to 0$ the point of closest approach of the R, ϵ curve to $(R_{00}, 0)$ comes nearer and nearer to that point. Clearly figure 4 is qualitatively very similar to that found by Jones *et al.*, and it seems that their solutions are a manifestation of a similar limit to the one described here.

5. Conclusions

In previous sections we have demonstrated the existence of an asymptotic solution to the problem of steady convection in the limit $\sigma \to 0$, at least in certain geometries. A major feature of this solution is the existence of a 'second critical Rayleigh number' $R_{00} > R_0$ which represents the lowest R at which irrotational advection of momentum can occur. The problem solved is formally nonlinear (although the equations in our particular geometry reduce to linear ones) and hence the solutions that are obtained are valid well outside the range of validity of normal perturbation theory.† This is not unusual in itself; what is new is that the analysis is able to distinguish between inertial and thermal nonlinearities, and to consider the first as dominant while the second remains only as a small perturbation. Our system is somewhat special in that the 'second critical Rayleigh number' R_{00} obtained is independent of σ. In more general geometries, we should expect that a transition from low to high Reynolds number could still occur, except that in general the inertial torques would vanish only away from the boundaries. The viscous boundary conditions would then have to be met by viscous boundary layers. We should therefore expect in general that $R_{00} \sim \sigma^{-\frac{1}{2}}$ or $\sigma^{-\frac{1}{3}}$, but that the characteristic 'kink' in the curve of Nusselt number *vs.* Rayleigh number would still occur. We should note, however, that the sort of limit described by Jones *et al.* and ourselves is most likely to be relevant to convection between free boundaries (as occurs in the sun) since then the system is free to choose its horizontal planform. Indeed, Spiegel (personal communication) plans to extend the work of Gough *et al.* (1975) to allow for this planform selection. This type of asymptotic problem occurs in other physical situations; in particular, the nonlinear α-effect dynamo models investigated by Malkus & Proctor (1975) and Proctor (1975) exhibit an essentially similar dichotomy in the basic eigenvalue problem depending on whether magnetic or viscous effects tend to zero most rapidly.

† While the analogous problem posed by Jones *et al.* cannot be solved by these simple methods, there is no doubt that *in principle* a similar perturbation expansion can be applied to their problem also.

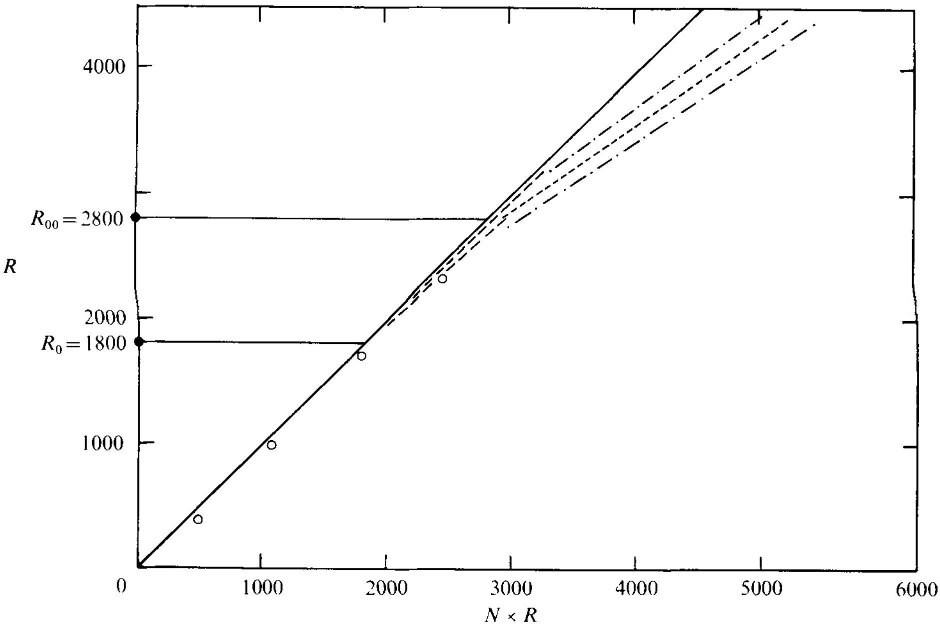

FIGURE 5. Results of an experiment by Rossby (1962) using mercury ($\sigma = 0{\cdot}026$) with a distance between the plates of 2 cm. The circles and central broken line give R as a function of NR in two different experimental configurations. For $R > R_0 \simeq 1800$ the fluid is in motion, so $N > 1$. The outer broken lines give an envelope for all experimental results in this region. At $R = 2800$ there seems to be a distinct break in the R, NR dependence.

A final question concerns the stability of the transition described here and the possibility of its being observed in the laboratory. It is generally considered that convection at low σ becomes unstable to time-dependent disturbances very near to the critical Rayleigh number R_0. Indeed, Clever & Busse (1974) have indicated that the value of R for the onset of time dependence tends to R_0 as $\sigma \to 0$. The experiments of Krishnamurti (1973), however, show that time-independent behaviour can persist for mercury ($\sigma \simeq 0{\cdot}026$) significantly beyond R_0; although she finds no evidence of the above transition, the scatter in her results is large. Rossby (1962) also carried out experiments on mercury, and found what appears to be a 'kink' in the curve of heat flux *vs.* Rayleigh number at $R \simeq 1{\cdot}5R_0$ (figure 5). Rossby's motions were time dependent and turbulent and so any analcgy is fraught with peril. We do consider, though, that the similarity between his curve and the one we predict for steady motions is not entirely fortuitous. The numerical calculations of Clever & Busse (1974) for convection between rigid boundaries shows no significant decrease in the Nusselt number as $\sigma \to 0$ as long as R is large enough. This would suggest strongly that inertial forces modify the flow in such a way that their effect on the body of the flow is not large. Our study shows one way in which this might be achieved: other ways, presumably involving time-dependent disordered motions, are beyond the scope of present analysis.

We believe that the transition that occurs in our particular geometry may well be observable in the laboratory. First, the observed weak dependence of the heat flux on the Prandtl number at high Rayleigh number is an indication of some basic order in the flow in which inertial forces are almost irrotational. Second, it has been observed above (§ 3) that for $R_0 < R < R_{00}$ the Reynolds number is a function of the Rayleigh number alone, and in particular is independent of σ. Jones *et al.* have noted that, for their geometry, the Reynolds number rises to about 30 near $R = R_{00}$. This is not very large, and it certainly seems plausible that the Reynolds number is low enough to permit time-independent behaviour up to $R = R_{00}$. Further, as R_{00} is approached, the flow streamlines become more and more circular, and one certainly feels that this pattern is *less* unstable than the linear eigensolution, whose streamlines are oval. We hope to carry out experiments to test this possibility.

The author wishes to record his gratitude to Dr C. A. Jones, Dr D. R. Moore and Dr N. O. Weiss for communicating to him their preliminary numerical results and for many fruitful discussions while their paper was being prepared. This work was undertaken while the author was a participant in the 1975 Geophysical Fluid Dynamics Summer Program, Woods Hole, Massachusetts. A first report of this work appears in the Fellowship Lectures for that year. Grateful thanks are due to all the Staff and Fellows, especially the Director, Professor W. V. R. Malkus, and to the National Science Foundation for support under Grant GA-32593.

Appendix. Asymptotic matching in the limit of small η

We seek solutions to (4.37) when η is small. If we first treat the outer problem, neglecting the viscous term, we can easily verify that

$$\chi_{11} = Q(s) + CJ_2(\alpha s), \tag{A 1}$$

where C is arbitrary as yet (and independent of any scaling). $Q(s)$ is a particular integral that is not well behaved at $s = 1$ since $J_1(\alpha) = 0$. Specifically,

$$\begin{aligned} Q(s) \sim\; & A(1-s)\ln(1-s) + O\{(1-s)^2 \ln(1-s)\} \\ & + A_1 + A_2(1-s) + O\{(1-s)^2\}, \end{aligned} \tag{A 2}$$

where $A = -\tfrac{1}{2}\alpha^2$ and A_1 and A_2 are real constants that depend linearly on A. R_{01} can be found without evaluating A_1 and A_2. Hence, near $s = 1$

$$\chi_{11} \sim A(1-s)\ln(1-s) + A_1 + A_2(1-s) + C[J_2(\alpha) + 2(1-s)J_2(\alpha)]. \tag{A 3}$$

For the inner problem, near $s = 1$, we define the new variables

$$\left.\begin{aligned} \xi &= \delta^{-1}(1-s), \\ \delta &= (\eta/2K\alpha^3 J_2(\alpha))^{\frac{1}{3}}, \quad \chi_{11} = \delta\tilde{\chi}(\xi). \end{aligned}\right\} \tag{A 4}$$

where

Then to leading order the inner problem becomes

$$i\tilde{\chi}_{\xi\xi\xi\xi} + \xi\tilde{\chi}_{\xi\xi} = A \tag{A 5}$$

with the boundary conditions $\tilde{\chi} = \tilde{\chi}_\xi = 0$ at $\xi = 0$. The solution for $\tilde{\chi}_{\xi\xi}$ can be written as

$$\tilde{\chi}_{\xi\xi} = Ai^{\frac{1}{3}}\,\mathrm{Gi}\,(\xi i^{\frac{1}{3}}) + \tilde{C}f(\xi i^{\frac{1}{3}}), \tag{A 6}$$

where $f(\xi)$ is the Airy function $\mathrm{Ai}(\xi)$ and

$$\mathrm{Gi}(\xi) \equiv \int_0^\infty \sin\left(\tfrac{1}{3}t^3 + \xi t\right) dt$$

(see Abramowitz & Stegun 1964, p. 448) Other solutions become exponentially large as $\xi \to \infty$ and are thus excluded by the matching conditions. $\tilde{C}$ is an arbitrary (and scale-independent) constant. Hence the full solution is

$$\tilde{\chi} = \int_0^{\xi} d\xi' \int_0^{\xi'} d\xi'' \tilde{\chi}_{\xi\xi}(\xi''), \tag{A 7}$$

and as $\xi \to \infty$ this becomes to leading order

$$\tilde{\chi} \sim A\xi \ln \xi + A_3 + A_4 \xi + \tilde{C}[C_1 + C_2 \xi] + O(\delta), \tag{A 8}$$

where A_3 and A_4 depend on A alone and C_1 and C_2 are independent of $\tilde{C}$. In order to match the inner and outer solutions, we must write (4.41) in terms of ξ. Thus, from (A 3),

$$\chi_{11} \sim \delta[A\xi \ln \xi + A\xi \ln \delta + A_1 \delta^{-1} + A_2 \xi + C_0[J_2(\alpha)\,\delta^{-1} + 2J_2(\alpha)\,\xi] + \delta C_\delta \ln \delta [J_2(\alpha)\,\delta^{-1} + 2J_2(\alpha)\,\xi]], \tag{A 9}$$

where C has been written as $C_0 + \delta \ln \delta C_\delta + o(\delta \ln \delta)$. Matching (A 8) with (A 9) then gives

$$A_1 + C_0 J_2(\alpha) = 0, \quad \tilde{C} = A \ln(\delta)/C_2 + O(1) \tag{A 10}$$

and therefore a term $A \ln(\delta)\, C_1/C_2$ appears in (A 8). It may easily be verified that there is no other term of this form and order in the higher-order expansion of the inner solution. Hence this term must be matched against a term in (A 9), so that clearly

$$C_0 = AC_1/C_2 J_2(\alpha) = 1{\cdot}035\alpha^2 i^{-\frac{1}{3}}/J_2(\alpha), \tag{A 11}$$

where C_1 and C_2 are easily found from the properties of Ai (ξ). There is no need to continue the matching further, C_δ being the leading-order term with non-zero imaginary part.

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